*Journal of Statistical Physics, Vol. 58, Nos. 5/6, 1990*

# The Dimension Spectrum of Axiom A Attractors

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*Received April 27, 1989; revision received September 6, 1989*

Some rigorous results on the dimension spectrum of expanding Markov maps of the interval are extended to Axiom A $C^2$ diffeomorphisms of a compact two-dimensional manifold.

**KEY WORDS:** Hyperbolic attractors; Hausdorff dimension spectrum; thermodynamic formalism.

## 1. INTRODUCTION

Recently some new ideas and techniques have been proposed[11] to characterize singularities of invariant measures $\mu$ and attractors on which these measures are concentrated. The (possibly) multifractal nature of an attractor $\Omega$ can be analyzed by partitioning it in a (large) number of atoms and looking at the atoms $I$ such that $\mu(I) \simeq |I|^{\alpha}$ ($|I|$ is the size of $I$). What can one say about the limit (i.e., for $|I| \to 0$) distribution of measures $\mu(\cdot)$? Following a statistical mechanics procedure, one writes the partition function

$$Z_n(\beta) = \sum_{\substack{I \in \text{partition of } \Omega \\ \text{in } 2^n \text{ atoms}}} \mu(I)^{\beta}$$

If the limit $\lim_{n \to \infty} (1/n) \log Z_n(\beta)$ exists and defines a regular function $F(\beta)$, then the distribution of the numbers $\mu(I)$ is related to the Legendre transform of $F(\beta)$, $f(\alpha) = \inf_{\beta} [\alpha\beta - F(\beta)]$: $f(\alpha)$ is precisely the Hausdorff dimension of the sets on which $\mu$ has a power law singularity $\alpha$.

In ref. 5 we showed how to prove some rigorous results on the existence, regularity, and "universality" of the function $f(\alpha)$ for expanding Markov maps of the interval. In this paper I discuss the extension of these

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results to Axiom A $C^2$ diffeomorphisms $f$ of a compact two-dimensional manifold $M$.

Suppose an invariant measure is given, supported on a compact $f$ invariant set $\Omega(f) \equiv \Omega$, where $f$ is transitive (see Section 2). The main requisite for Axiom A is for $\Omega$ to be hyperbolic (see Section 2). Hyperbolicity means dynamics that contracts in some directions, stretches in the others, to stretch and then (because of compactness) to fold objects. Orbits separate exponentially fast (along expanding directions) at an exponential rate which is uniform on $\Omega$. Thus, combined with compactness, the presence of positive exponents generates "chaos" and, even if the dynamics contracts the initial volume, makes possible motion that is "exponentially unstable" or "chaotic" on the attractor.(7,25) A natural measure for an Axiom A attractor $\Omega$ would be a measure which describes the statistics of orbits originating in a *neighborhood* of $\Omega$ and which is carried by the limit set of these orbits, i.e., by the attractor. Define the Sinai–Ruelle–Bowen measure $\mu$ as the limit of the sequence of the images, via $f^n$, of the Lebesgue measure concentrated on a sufficiently small neighborhood of $\Omega$.(19,23) It is rather intuitive(25) that this measure can be disintegrated along unstable manifolds(18) by conditional probabilities absolutely continuous to Lebesgue, and most important is that this measure describes the asymptotic distribution of trajectories originating in a set of positive *Lebesgue* measure(23) (see Section 2). Moreover, the "requirement" on conditional probabilities to be smooth has here, via symbolic dynamics, the natural equivalence in statistical mechanics for $\mu$ to be the (unique) Gibbs state associated to the Hölder continuous potential involving the Jacobian of unstable directions.(4,24,27)

Nevertheless, $\mu$ may be singular in the stable directions and in general(27) it is.

Whenever the unstable manifold has codimension one, the possibly complicated structure of attractors can be investigated by one-dimensional techniques, since the decomposition theorem referred to(19,21,25,27) shows that the "interesting part" (that is, the possible singularities) of the measure $\mu$ can be read off on the transverse measure. This remark, together with a natural hypothesis of transitivity of the system(26) and the well-known distortion lemma, allows the extension of one-dimensional techniques used in ref. 5 to characterize singularities of measures carried by Axiom A attractors.

I work on a space which is in fact a quotient space $\Omega/\xi$ with respect to the (local) unstable foliation $\xi$ and show the existence "*uniformly*" on $\Omega$ of

$$F(\beta) = \lim_{n \to \infty} (1/n) \log \sum_{I \in \{A_n\}} v_\xi(I)^\beta \tag{1.1}$$

where $\nu_\xi(\cdot)$ is the "transverse measure" and $\{A_n\}$ is a suitable "uniform partition" of a local stable manifold.

The regularity of $F$ follows from the fact that $F(\beta)$ turns out to be the inverse function of another partition function $G$ [i.e., $G(\beta, F(\beta)) = 0$], whose existence and regularity are well-known statistical mechanics results[13,14,24,27]

$$G(x, y) = \lim_{n \to \infty} (1/n) \log \sum_{A \text{ s.t. } \Delta_A \in V_{i=0}^{n-1} f^i(R^{(0)})} \nu_\xi(A)^x \, |A|_\xi^y, \qquad \forall x, y \in \mathbb{R} \tag{1.2}$$

where $R^{(0)}$ is a Markov partition of $\Omega$,[4] and $\Delta_A$ is the $u$-subrectangle associated to $A$ with respect to $R^{(0)}$,[4] and $|\cdot|_\xi$ means the "transverse volume" with respect to $\xi$, that is, the length of $A$ measured along the unstable foliations.

The regularity of $F(\beta)$ enables one to apply large-deviation theorems to prove that its Legendre transform $f(\alpha)$ is the Hausdorff dimension of the sets

$$B^{\pm\alpha} = \left\{ x \in \Omega \text{ s.t. } \lim \begin{pmatrix} \sup \\ \inf \end{pmatrix}_{|I_x|_\xi \to 0} \frac{\log \nu_\xi(I_x)}{\log |I_x|_\xi} = \alpha \right\}$$

whose characterization is enough to capture the singularities (in the above described sense) of $\mu$ (see ref. 5).

Similar results on $f(\alpha)$ based entirely on the partition function $G$ have been obtained by Rand[20] and Gundlach.[10]

Sections 2 and 3 recall the properties of the Sinai–Ruelle–Bowen measure for the Axiom A attractors and prove some distortion lemmas. The proof of the existence of the limits (1.1) ("uniform free energy") and (1.2) ("dynamic free energy") occupies respectively Sections 4 and 5. Finally, Section 6 is devoted to the study of the relations between these two free energies.

## 2. ON AXIOM A DIFFEOMORPHISMS

Let $f: M \to M$ be a $C^2$-diffeomorphism of a compact two-dimensional Riemannian manifold $M$. Let $f$ satisfy Axiom A [i.e., the nonwandering set $\Omega(f)$ is hyperbolic and it can be described as the closure of periodic points]. It is known that $\Omega(f)$ decomposes in a finite union of closed, disjoint, invariant sets $\Omega_i$ ("basic sets") such that $f \mid \Omega_i$ is topologically transitive.[26] Hence it is rather natural to restrict our analysis to the case where $f$ is transitive on $\Omega$. Recall that $\Omega$ has a hyperbolic structure if there is a continuous splitting of the tangent space in a direct sum $TM = E^s \oplus E^u$, invariant under $Df$, so that $Df: E^s \to E^s$ is contracting and

$Df: E^u \to E^u$ is expanding. Moreover, let $W^u(\Omega)$, $W^s(\Omega)$ be the global unstable and stable manifolds, and $W^u(x)$ and $W^s(x)$ be the global unstable and stable manifolds at $x$ and $W^s_l(x)$ and $W^u_l(x)$ the local stable and unstable local manifolds at $x$ of size $l$.[4,18] I shall denote by $\lambda_u^{-1}(x)$ $[\lambda_s^{-1}(x)]$ the length of the image of the unitary vector $e_u(x)$ in $E^u_x$ $[e_s(x)]$ under the action of $Df^{-1}[Df]$. I denote by $m$ the volume induced by the Riemannian metric on $M$.

The Arnold toral automorphism[3] is a nice example which provides the simplest picture of what Axiom A looks like. Here $M$ is the two-dimensional torus $T^2 = R^2/Z^2$ and $f$ is the linear automorphism $(x, y) \to (2x + y, x + y) \bmod Z^2$, $(x, y) \in R^2$. The $Df$ is everywhere equal to the matrix $\left(\begin{smallmatrix}2 & 1\\ 1 & 1\end{smallmatrix}\right)$, which has eigenvalues $\lambda_1 \equiv \lambda = (3 + \sqrt{5})/2$ and $\lambda_2 = \lambda^{-1}$ with eigenvectors $v_1 = (1, 1 - \lambda)$ and $v_2 = (1, 1 - \lambda^{-1})$. The contractive and expansive subspaces $E^s(x)$ and $E^u(x)$ do not depend on $x$; they are the one-dimensional spaces spanned by $v_1$ and $v_2$. Because of linearity, the global stable and unstable manifolds are given by $W^u(x) = E^u$ and $W^s(x) = E^s$, and these straight lines emerging from the point $x$ with irrational slope $(-1 + \sqrt{5})/2$ thus wind densely around the torus. The whole torus is (uniformly) hyperbolic, i.e., (definition) $f$ is Anosov. Anosov diffeomorphisms satisfy always Axiom A. Here it is easy to see that all points of $T^2$ with rational coordinates are periodic. I call this the conservative case: the whole torus may be considered as a (chaotic) attractor. It does not display a fractal structure. The Lebesgue measure is conserved $(\lambda_1 \lambda_2 = 1)$, as well as for any automorphism of $T^2$ (see ref. 1): here $\mu = m$ and this is the trivial case. In general, for Anosov diffeomorphisms $\mu \neq m$ (see Section 4), but I am most interested in dissipative cases, that is, $\mu(\Omega) = 1$ and $m(\Omega) = 0$.

Let us come back to the general situation. Let $\Omega$ be an Axiom A attractor; then it is known that there exists a unique $f$-invariant, ergodic measure $\mu$ such that the following holds.

**Theorem 2.1.**[23,27] 1. For $m$-almost every $x \in W^s(\Omega)$,

$$\lim(1/n) \sum_{i=0}^{n-1} g(f^i(x)) = \int_\Omega g \, d\mu \tag{2.1}$$

$g$ any continuous function: $M \to R$.

2. $\mu$ can be disintegrated along the unstable foliations by conditional probabilities absolutely continuous with respect to the Lebesgue measure. Let $\xi$ be (the partition of $\Omega$; see ref. 19) $= \bigcup_{x \in \Omega} W^u_{\text{loc}}(x)$, and let $c_\xi(x)$ be

the element of $\xi$ which contains $x$; then to any $x \in \Omega$ we associate $c_\xi(x)$ via the canonical projection

$$H:\quad \Omega \to \Omega/\xi$$
$$x \to c_\xi(x)$$

Let $A \in \Omega$; then

$$\mu(A) = \int_{\Omega/\xi} \mu_{c_\xi}(A \cap c_\xi)\, d\mu_\xi \tag{2.2}$$

where (a) $\int_{c_\xi(x)} d\mu_{c_\xi(x)}(z) = 1$; (b) if $l$ is the Lebesgue measure on $W^u$, we have

$$\mu_{c_\xi(x)}(A \cap c_\xi(x)) = \frac{\int_{c_\xi(x) \cap A} \rho_x(z)\, dl_x(z)}{\int_{c_\xi(x)} \rho_x(z)\, dl_x(z)}$$

and $\rho_x(z) = \prod_{i=1}^{\infty} \lambda_u^{-1}(f^{-i}(x))/\lambda_u^{-1}(f^{-i}(z))$ is a continuous function of $z \in W^u(x)$, $\forall x \in \Omega$.

3. $\mu$ maximizes the expression

$$h_\mu + \int -\log \lambda_u^{-1}(x)\, d\mu(x) \tag{2.3}$$

where $h$ is the entropy; this maximum is 0.[18,23]

Moreover, recall the following results.

**Theorem 2.2.**[4] Any basic set $\Omega$, for an Axiom A diffeomorphism, does have a Markov partition of (arbitrary) small diameter.

**Lemma 2.3.**[2,18]

$$\lambda_s^{-1}(x) = \langle e_s(f(x)), Df(e_s(x))\rangle \tag{2.4}$$

is a Hölder continuous function on $\Omega$.

Define $\bar{\lambda}^{-1}$ and $\lambda$ by

$$\begin{aligned} &0 < \bar{\lambda}^{-1} \leqslant \lambda_s^{-1}(x) \leqslant \lambda < 1 \\ &0 < \bar{\lambda}^{-1} \leqslant \lambda_u^{-1}(x) \leqslant \lambda < 1, \qquad \forall x \in \Omega \end{aligned} \tag{2.5}$$

I prove now a distortion lemma useful later.

**Lemma 2.4.** Let $t, t' \in \Omega$; let $L_s(t)$ be a small arc on a local stable manifold $W^s(t)$ and $L_s(t')$ the corresponding arc obtained by the projection

$\pi$ along the unstable manifolds on $W^s(t')$. Then there exists a positive constant $c$ and an integer $\bar{q}$ which is the smallest integer such that $|f^{-\bar{q}}(L_s(t))| = O(1)$, such that $\forall q \leqslant \bar{q}$,

$$c^{-1}\frac{|L_s(t)|}{|L_s(t')|} \leqslant \frac{|f^{-q}(L_s(t))|}{|f^{-q}(L_s(t'))|} \leqslant c\,\frac{|L_s(t)|}{|L_s(t')|} \tag{2.6}$$

*Proof.* Let $t_1$, $t_2$, $t'_1$, $t'_2$ be the boundary points of $L_s(t)$, $L_s(t')$. Let $ds$ denote a coordinate on $W^s$ and if $x_j = f^j(x_0) = f^j(f^{-q}(t))$ and $y_j = f^j(y_0) = f^j(f^{-q}(\tau))$, where $\tau$ is a point in $\Omega$ sufficiently near to $t$, then

$$\begin{aligned} |L_s(t)| &= \int_{t_1}^{t_2} ds = \int_{f^{-q}(t_1)}^{f^{-q}(t_2)} dx\, \langle Df^q(e_s(x)), Df^q(e_s(x)) \rangle^{1/2} \\ &= \int_{f^{-q}(t_1)}^{f^{-q}(t_2)} dx_0 \prod_{j=0}^{q-1} \lambda_s^{-1}(x_j) \\ &= \int_{f^{-q}(t_1)}^{f^{-q}(t_2)} dx_0 \prod_{j=0}^{q-1} \lambda_s^{-1}(y_j)\, \frac{\lambda_s^{-1}(x_j)}{\lambda_s^{-1}(y_j)} \end{aligned} \tag{2.7}$$

Using the Hölder continuity of $\lambda_s^{-1}$, we get

$$\begin{aligned} \prod_{j=0}^{q-1} \frac{\lambda_s^{-1}(x_j)}{\lambda_s^{-1}(y_j)} &= \exp \sum_{j=0}^{q-1} [\log \lambda_s^{-1}(f^{-q+j}(t)) - \log \lambda_s^{-1}(f^{-q+j}(\tau))] \\ &\leqslant \exp \sum_{j=0}^{q-1} c\lambda^{(q-j)\alpha}\, d(x, y)^\alpha \\ &\leqslant \exp\left(\frac{c}{1+\lambda}\right) \end{aligned} \tag{2.8}$$

It follows that

$$c^{-1} \leqslant \prod_{j=0}^{q-1} \frac{\lambda_s^{-1}(x_j)}{\lambda_s^{-1}(y_j)} \leqslant c \tag{2.9}$$

Hence

$$c^{-1} \prod_{j=0}^{q-1} \lambda_s^{-1}(f^{-q+j}(\tau)) \leqslant \frac{|L_s(t)|}{|f^{-q}(L_s(t))|} \leqslant c \prod_{j=0}^{q-1} \lambda_s^{-1}(f^{-q+j}(\tau)) \tag{2.10}$$

By applying the same estimates to $L_s(t')$ and choosing the same point $\tau$, one gets the result.

## 3. THE PARTITION FUNCTION $Z(\beta)$

Let $R^{(0)} = \{R_0^{(0)}, R_1^{(0)},..., R_d^{(0)}\}$ be a Markov partition of $\Omega$, and let $\xi$ be the partition

$$\xi_{R^{(0)}} = \bigcup_{x \in \Omega} (R^{(0)} \cap W^u(x))$$

Let $W^s_{l_0}(y)$ be the local stable manifold at $y \in \Omega$, $l_0 = \max \operatorname{diam} R_i^{(0)}$. Let $W^s(y, R_i^{(0)}) = W^s_{l_0}(y) \cap R_i^{(0)}$. Let $\{A_n\}$ be a "uniform partition" of $W^s(y, R_i^{(0)})$ (i.e., a partition into $2^n$ atoms of the same length). Let $I \in A_n$, let $\Delta_I$ be the $u$-subrectangle of $R_i^{(0)}$ associated to $I$.[4] Let $H$ be the canonical projection $H: \Omega \to \Omega/\xi_{R^{(0)}}$ and $\tilde{\Delta}_I = H\Delta_I$.

**Definition 3.1.** Define the transverse measure $v_{\xi_{R^{(0)}}}(I)$ of $I$ with respect to the partition $\xi_{R^{(0)}}$ a

$$v_{\xi_{R^{(0)}}}(I) \equiv \mu_{\xi_{R^{(0)}}}(\tilde{\Delta}_I) \tag{3.1}$$

where by the decomposition theorem[21]

$$\begin{aligned}
\mu(\Delta_I) &= \int_{\Omega/\xi_{R^{(0)}}} \mu_{c_{\xi_{R^{(0)}}}}(\Delta_I \cap c_{\xi_{R^{(0)}}})\, d\mu_{\xi_{R^{(0)}}} \\
&= \int_{\tilde{\Delta}_I \subset \Omega/\xi_{R^{(0)}}} \mu_{c_{\xi_{R^{(0)}}}}(c_{\xi_{R^{(0)}}})\, d\mu_{\xi_{R^{(0)}}} \\
&= \int_{\tilde{\Delta}_I} d\mu_{c_{\xi_{R^{(0)}}}} \\
&= \mu_{\xi_{R^{(0)}}}(\tilde{\Delta}_I)
\end{aligned} \tag{3.2}$$

I list now some useful properties of $v_{c_{\xi_{R^{(0)}}}}$.

**Lemma 3.2.** Let $\mu$ be the Sinai–Ruelle–Bowen measure; suppose that there exists a constant $c$ such that $\mu(f(A)) \leqslant c\mu(A)$, for any measurable set $A$. We have (all constants will be denoted by the same letter $c$) the following results.

1. Let $R'$, $R''$ be two Markov partitions of $\Omega$. Let $I \in \{A_n\}$. Then

$$c^{-1} v_{\xi_{R'}}(I) \leqslant v_{\xi_{R''}}(I) \leqslant c v_{\xi_{R'}}(I)$$

2. $f^{-1}\xi$ is a partition of $\Omega$ and $f^{-1}\xi > \xi$ [that is, $c_\xi(x) \supset c_{f^{-1}\xi}(x)$], $f\xi$ is a partition of $\Omega$, and $f\xi < \xi$.

3. By 2, the $f$ invariance of $\mu$, and the Markov property we have ("quasi-invariance")

$$c^{-1}\nu_\xi(f^{-1}(I))\leqslant \nu_\xi(I)\leqslant c\nu_\xi(f^{-1}(I))$$

Moreover, we have the following "distortion lemma" for the measure $\nu_\xi$.

**Lemma 3.3.** Let $I\subset J$ be small arcs on a local stable manifold, and let $q$ be the smallest integer such that $|f^{-q}(J)|=O(1)$; then there exists a finite constant $c$ such that

$$c^{-1}\frac{\nu_\xi(f^{-q}(I))}{\nu_\xi(f^{-q}(J))}\leqslant\frac{\nu_\xi(I)}{\nu_\xi(J)}\leqslant c\frac{\nu_\xi(f^{-q}(I))}{\nu_\xi(f^{-q}(J))} \tag{3.3}$$

*Proof.* This is a simple consequence of the invariance of $\mu$ and the distortion Lemma 2.4. Let $\Delta_I$ and $\Delta_J$ be the $u$-subrectangles associated to $I$ and $J$, and $l$ the Lebesgue measure on $W^u$. Then

$$\begin{aligned}\frac{\mu(f^{-q}(\Delta_I))}{\mu(f^{-q}(\Delta_J))}&=\frac{\int_{\Omega/\xi_{R(0)}}\mu_{c_{\xi_{R(0)}}}(f^{-q}(\Delta_I)\cap c_{\xi_{R(0)}})\,d\mu_{\xi_{R(0)}}}{\int_{\Omega/\xi_{R(0)}}\mu_{c_{\xi_{R(0)}}}(f^{-q}(\Delta_J)\cap c_{\xi_{R(0)}})\,d\mu_{\xi_{R(0)}}}\\&\leqslant o(1)\frac{\nu_\xi(f^{-q}(I))}{\nu_\xi(f^{-q}(J))}\frac{l(f^{-q}(\Delta_I))}{l(f^{-q}(\Delta_J))}\\&\leqslant o(1)\frac{\nu_\xi(f^{-q}(I))}{\nu_\xi(f^{-q}(J))}\frac{l(\Delta_I)}{l(\Delta_J)}\end{aligned}$$

Also,

$$\begin{aligned}\frac{\mu(\Delta_I)}{\mu(\Delta_J)}&=\frac{\int_{\Omega/\xi_{R(0)}}\mu_{c_{\xi_{R(0)}}}(\Delta_I\cap c_{\xi_{R(0)}})\,d\mu_{\xi_{R(0)}}}{\int_{\Omega/\xi_{R(0)}}\mu_{c_{\xi_{R(0)}}}(\Delta_J\cap c_{\xi_{R(0)}})\,d\mu_{\xi_{R(0)}}}\\&\leqslant o(1)\frac{\nu_\xi(I)}{\nu_\xi(J)}\frac{l(\Delta_I)}{l(\Delta_J)}\end{aligned}$$

Reverse inequalities are proven similarly, and by the invariance of the measure $\mu$ we get the result.

## 4. PROOF OF THE EXISTENCE OF $F(\beta)$

**Definition 4.1.** Let $f$ be a $C^2$ expanding Markov map of the interval, and $\mu$ an $f$ invariant measure. We say that a uniform partition $\{A_n\}$ of the interval {i.e., a partition with atoms of uniform length $2^{-n}$) is $(\theta, l)$-regular if, letting for any $I\in\{A_n\}$, $q=q_I$ be the smallest integer such that $f^q(I) \cap \partial R \neq \emptyset$ (where $R$ is a Markov partition), then $|f^q(I)| > l$ and $\mu(f^q(I)) > \theta$, with $\theta$ and $l$ positive numbers *independent* of $n$.

**Proposition 4.2.**[5] Let $f$ be a $C^2$ expanding Markov map of the interval, and $\mu$ an $f$ invariant measure. Let $\{A_n\}$ be a partition of the interval with atoms of uniform length $2^{-n}$. Then, if $n$ is large enough, there exists a $(\theta, l)$-regular partition $\{\tilde{A}_n\}$ of the interval with atoms $I$ such that $|I| = O(1)\, 2^{-n}$.

There are, of course, some consistency relations among $\theta$, $l$, $\lambda$, $n$, and the size of the Markov partition to be satisfied; see ref. 5.

**Corollary 4.3.** There exists a $(\theta, l)$-regular partition $\{\tilde{A}_n\}$ of $W^s(y, R_i^{(0)})$, that is, letting $q = q_I$ be the smallest integer such that $f^{-q}(I) \cap \partial^u R_i^{(0)} \neq \emptyset$, then $|f^{-q}(I)| > l$ and $\nu_\xi(f^{-q}(I)) > \theta$, and $|I| = O(1)\, 2^{-n}$, $n$ large enough.

The requirement for $\{A_n\}$ to be a regular partition is not only a technical one to avoid uncontrollable contributions to the partition function at the thermodynamic limit, but it accounts for the structure of the support of the measure.[5]

**Theorem 4.4.**[5] Let $f$ be a $C^2$ expanding Markov map of the interval, $\{A_n\}$ a uniform regular partition of the interval, $n$ large enough, and $\mu$ an $f$ invariant measure such that if $I$ and $J$ are atoms of $\{A_n\}$ in the same Markov rectangle; then

$$\frac{\mu(I)}{\mu(J)} = \frac{\mu(f(I))}{\mu(f(J))}$$

Let

$$Z_n(\beta) = \sum_{I \in \{A_n\}} \mu(I)^\beta$$

Then $\forall \beta \in R$ the following limit exists:

$$\lim_{n \to \infty} (1/n) \log Z_n(\beta) = F(\beta) \tag{4.1}$$

The existence of the thermodynamic limit (4.1) is shown by an argument of subadditivity of the partition function (using the analogues on the interval of Lemmas 2.4 and 3.4), complicated by the presence of the boundaries of the Markov partition.

I shall prove the following result.

**Theorem 4.5.** Let $n$ be large enough; let

$$Z_n(\beta) = \sum_{I \in \{A_n\}} \nu_{c_{\xi_{R^{(0)}}}}(I)^\beta \tag{4.2}$$

where $\{A_n\}$ is a uniform regular partition of $W^s(y, R_i^{(0)})$. Then the

$$\lim_{n \to \infty} (1/n) \log Z_n(\beta) = F(\beta) \tag{4.3}$$

exists uniformly with respect to $y \in \Omega$ and is independent of $y$.

**Definition 4.6.** I say that a partition $\{A_n\}$ is of the order $n$ if its atoms have a length $o(1)\, 2^{-n}$. I say that the partition functions $Z_n(\beta)$ and $Z_{n+l}(\beta)$ associated respectively to $\{A_n\}$ and $\{A_{n+l}\}$ are equivalent partition functions if there exists a sequence $c_n$ with $\lim_{n \to \infty} [(\log c_n)/n] = 0$, such that

$$c_n^{-1} Z_n(\beta) \leqslant Z_{n+l}(\beta) \leqslant c_n Z_n(\beta)$$

I am now ready to prove the following result.

**Lemma 4.7: Reduction to the One-Dimensional Case.** Let $y, z \in R_i^{(0)}$; let $\{A_n\}$, $n$ large enough, be a uniform regular partition of $W^s(R_i^{(0)}, y)$. We construct a partition $\{A'_{n'}\}$ of $W^s(R_i^{(0)}, z)$ by projecting by $\pi$ along the local unstable manifolds the elements of $\{A_n\}$. We compare the element $I \in \{A_n\}$ and $\pi I = I'$ with the corresponding element of $\{A'_{n'}\}$. Observing that $I$ and $I'$ do meet $\partial^u R^{(0)}$ under the action of $f^{-1}$ at the *same* number of iterations $q$, by the distortion lemma and the transversality of the measure, we obtain that if $\{A_n\}$ is $(\theta, l)$-regular, then there exist constants $c_1$, $c_2$, and $c_3$ such that $\{A'_{n'}\}$ is $(\theta', l')$-regular with $c_1^{-1} l \leqslant l' \leqslant c_1 l$, $\theta = \theta'$, and $l'/c_2 \leqslant |I|/|I'| \leqslant c_2/l$, so that the partition $\{A'_{n'}\}$ is of order $n'$ with

$$n + \log(l/c_3) \leqslant n' \leqslant n + \log(c_3/l)$$

Observing that $\nu_\xi(I) = \nu_\xi(I')$ since $\tilde{\Delta}_I = \tilde{\Delta}_{I'}$, it follows that if we define $Z_n^y(\beta)$ by

$$Z_n^y(\beta) = \sum_{\substack{I \in \{A_n\} \\ \text{partition of } W^s(R_i^{(0)}, y)}} \nu_\xi(I)^\beta \tag{4.4}$$

(and similarly for $z$), then $Z_n^y$ and $Z_{n'}^z$ are equivalent partition functions. We can therefore consider the class

$$Z_n^{R_i^{(0)}}(\beta) = \sum_{\substack{I \in \{A_n\} \text{ partition of} \\ W^s(R_i^{(0)}, y),\ \text{any } y \in R_i^{(0)}}} \nu_\xi(I)^\beta \tag{4.5}$$

Let now $q = q_{ij}$ be the smallest integer $q$ such that $f^{-q}(W^s(R_i^{(0)}, y)) \cap R_j^{(0)} \neq \emptyset$; let $\mathscr{L}$ be the sub arc of $W^s(R_i^{(0)}, y)$ such that $f^{-q}(\mathscr{L}) = f^{-q}(W^s(R_i^{(0)}, y)) \cap R_j^{(0)}$. Observe that, letting

$$Z_n^{\mathscr{L}}(\beta) = \sum_{\substack{I \subset \mathscr{L},\, I \in \{A_n\} \\ \text{partition of } W^s(R_i^{(0)}, y)}} v_\xi(I)^\beta$$

$Z_n^{\mathscr{L}}$ and $Z_n^{R_i^{(0)}}$ are equivalent partition functions. Now write $Z_n^{\mathscr{L}}$ equivalent to $Z_n^{R_i^{(0)}}$ and consider the images of the atoms of $Z_n^{\mathscr{L}}$ under $f^{-q}$, $q = q_{ij}$, in $R_j^{(0)}$. These images are atoms of a partition function of the order of $n + q \log 1/\lambda$, which is equivalent to any partition $Z_n^{R_j^{(0)}}$ up to a distortion $O(1)\, 2^{\log(c_2/l)}$. So we have obtained that there exist constants $C_0$, $c_4$ such that

$$C_0^{-1} Z_{n+p}^{R_j^{(0)}}(\beta) \leqslant Z_n^{R_i^{(0)}}(\beta) \leqslant C_0 Z_{n+p}^{R_j^{(0)}}(\beta) \tag{4.6}$$

with $\log[\lambda^{-q}(l/c_4)] \leqslant p \leqslant \log[\lambda^{-q}(c_4/l)]$.

Summarizing, I have used the *transversality* of the measure to show that corresponding (i.e., by projection) intervals have the same measure, the *distortion* lemma to control their relative lengths, and *transitivity* to paste together partitions associated to different Markov rectangles.

By Lemma 4.7, and using Lemmas 2.4 and 3.4 to prove subadditivity, as in Theorem 4.4, Theorem 4.5 follows.

*Remark.* Free energy of absolutely continuous measures.

In general $\beta \to F(\beta)$ for an absolutely continuous measure is not a trivial curve. However, we have the following result.

**Theorem 4.8.**[2,27] Let $f\colon M \to M$ be a $C^2$ transitive Anosov diffeomorphism. Let $\mu$ be the Sinai–Ruelle–Bowen measure. Then generically $\mu$ is singular with respect to the Lebesgue measure. If $f$ leaves invariant a measure $\tilde{\mu}$ absolutely continuous with respect to the Lebesgue measure, then $\tilde{\mu} = \mu$ and $\mu$ is absolutely equivalent to $m$.

**Corollary 4.9.** If $\mu \ll m$, then $F(\beta)$ is trivial.

## 5. SYMBOLIC DYNAMICS

Let $m$ be the Lebesgue measure on $M$, and $\mu_0$ the maximum entropy measure on $\{1,..., d\}^{\mathbb{Z}}$. Let $R_n^+ = V_{j=0}^{n-1} f^j(R^{(0)})$ the Markov partition which $s$-refines $R^{(0)}$.[4] Let $A$ be a sub arc of $W^s(R_i^{(0)}, y)$ (any $y \in \Omega$), such that its associated $u$-subrectangle $\Delta_A$ (see ref. 4) is an element of $R_+^n$. I define the "transverse volume" of $A$ as $|A|_\xi \equiv m(\Delta_A)$. Starting with the volume $m$, one can construct via the symbolic dynamic a sequence of measures $m_{0n}$ defined by their conditional probabilities

$$\frac{m_{0n}(\bar{\omega}_{-n},..., \bar{\omega}_n \mid \omega_i, |i| > n)}{m_{0n}(\bar{\bar{\omega}}_{-n},..., \bar{\bar{\omega}}_n \mid \omega_i, |i| > n)} = \exp\left[\sum_{i=0}^{n-1} A_u(\tau^i(\bar{\omega})) - A_u(\tau^i(\bar{\bar{\omega}})) + A_s(\tau^{-i}(\bar{\omega})) - A_s(\tau^{-i}(\bar{\bar{\omega}}))\right] \quad (5.1)$$

where $A_u(\cdot) = -\log \lambda_u^{-1}(\pi(\cdot))$ and $A_s(\cdot) = -\log \lambda_u^{-1}(\pi(\cdot))$ are Hölder continuous functions, $\pi: s\{\mathbf{1},..., d\}^{\mathbb{Z}} \to \Omega$ with $\pi(\omega) = \bigcap_{j=-\infty}^{+\infty} f^{-j} R_{\omega_j}^{(0)}$, $\tau$ is the shift on $\{\mathbf{1},..., d\}^{\mathbb{Z}}$, and $\bar{\omega} = (..., \bar{\omega}_{-n},..., \bar{\omega}_n, \omega_{n+1},...)$, $\bar{\bar{\omega}} = (..., \bar{\bar{\omega}}_{-n},..., \bar{\bar{\omega}}_n, \omega_{n+1},...)$ agree on the sites $i$ with $|i| > n$.

**Lemma 5.1.**[27] Let $A$ be a Hölder continuous function on $\{\mathbf{1},..., d\}^{\mathbb{Z}}$. Then $A(\omega) = \text{const} + \sum_{n>0} \psi_n(\omega_{-n},..., \omega_n)$, where $\psi_n$ are Hölder continuous cylindric functions, and there exist $c$ and $k > 0$ such that $|\psi_n| \leqslant ce^{-kn}$.

The above sequence then leads to a measure *formally* given by

$$dm_0 = Z^{-1} \exp\left\{\sum_{n=0}^{\infty} [A_u(\tau^n(\omega)) + A_s(\tau^{-n}(\omega))]\right\} d\mu_0 \quad (5.2)$$

where $\mu_0$ is the maximum entropy measure[27] and $Z$ is a normalization factor. Recall the following result.

**Lemma 5.2.**[27] Let $A$ be a Hölder continuous function on $\{\mathbf{1},..., d\}^{\mathbb{Z}}$. Then $A(\omega) = B(\omega^+) - u(\tau\omega) + u(\omega)$, where $B$ is a Hölder continuous function which depends only on the $\omega_j$, $j \geqslant 0$, positive sites, denoted $\omega^+$, and $u$ is a Hölder continuous function $u(\omega) = \sum_{n>0} \sum_{i=0}^{n-1} \psi_n(\tau^{n-i}\omega)$.

The "transverse volume" can be then coded in $m_0^+$, the restriction to $\mathbb{Z}^+$ of $m_0$:

$$\begin{aligned} dm_0^+ &= Z^{-1} \exp\left[\sum_{n=0}^{\infty} B_u(\tau^n(\omega^+))\right] \\ &\quad \times \left(\sum_{\omega^-} \exp\left[\sum_{n=-\infty}^{0} B_s(\tau^n(\omega^-))\right] \exp[u_u(\omega) + u_s(\omega)]\right) d\mu_0 \\ &= f(\omega)\, Z^{-1} \exp\left[\sum_{n=0}^{\infty} B_u(\tau^n(\omega^+))\right] d\mu_0 \end{aligned} \quad (5.3)$$

It follows that $m_0^+ \simeq \lambda^+$, the Gibbs measure on $\mathbb{Z}^+$ with rapidly decreasing potential $\Psi_X(\omega_X) = B_u(\omega_0,..., \omega_n, 0, 0,...) - B_u(\omega_0,..., \omega_{n-1}, 0, 0,...)$ if $X = [0,..., n]$ and 0 otherwise. By applying this coding to $m(\cdot) = |\cdot|_\xi^y$, we obtain that $\forall x, y \in \mathbb{R}$ the following limit exists:

$$\lim_{n \to \infty} (1/n) \log \sum_{A \text{ s.t. } \Delta_A \in R_+^n} \nu_\xi(A)^x |A|_\xi^y = G(x, y) \tag{5.4}$$

(see refs. 14, 24, and 28; see also ref. 22).

## 6. RELATIONS BETWEEN SINGULARITIES AND LIAPUNOV SPECTRA

Let $f: M \to M$ be a $C^2$ Axiom A diffeomorphism, let $\mu$ be the Sinai–Ruelle–Bowen measure, and let $\nu_\xi$ be the "transverse" measure defined in (3.1). Here I rephrase some known results in ergodic theory.

**Proposition 6.1.**

$$G(\beta, F(\beta)) = 0 \tag{6.1}$$

*Proof.* See ref. 5.

Define $\lambda(y) = (\partial G/\partial y)(\mathbf{1}, y)$; it follows that the characteristic (Liapunov) exponent with respect to the stable direction is $\lambda_2 = (\partial G/\partial y)(\mathbf{1}, 0)$ and $G(\mathbf{1}, y)$ is the Legendre transform of $\Phi(\Lambda)$, the "Liapunov spectrum."[6,9]

**Corollary 6.2.**[16]

$$HD(\nu_\xi) = -\frac{h(\nu_\xi)}{\lambda_2} \tag{6.2}$$

*Proof.* By differentiation from Proposition 6.1, recalling that the metric entropy of $\nu_\xi$ is $h(\nu_\xi) = (\partial G/\partial x)(\mathbf{1}, 0)$ and $\alpha(\mathbf{1}) = (dF/d\beta)(\mathbf{1})$ is the Hausdorff dimension of the measure $\nu_\xi$.

**Proposition 6.3.**

(a) $HD(\mu) = 1 + HD(\nu_\xi)$.

(b) $h(\mu) = h(\nu_\xi)$.

*Proof.* We have that ($\mu$ a.e. $x$) $HD(\mu) = \lim_{\varepsilon \to 0} [\log \mu(B(x, \varepsilon))/\log \varepsilon]$, where $B(x, \varepsilon) = \{y \in M \text{ s.t. } d(x, y) \leqslant \varepsilon\}$ (see ref. 29). According to Theorems 2.1 and 2.2, we have $\mu(B(x, \varepsilon)) = \int_{\Omega/\xi} \mu_{c_\xi(y)}(B(x, \varepsilon) \cap c_\xi(y))\, d\mu_\xi(y)$, which gives $c^{-1}\nu_\xi(I(x, \varepsilon))\varepsilon \leqslant \mu((B(x, \varepsilon)) \leqslant c\nu_\xi(I(x, \varepsilon))\varepsilon$, where $I(x, \varepsilon) = B(x, \varepsilon) \cap W_\varepsilon^s(x)$, and $c$ is a constant. This proves (a). Equality (b) follows by observing that $h(\mu)$ is the (metric) entropy of the endomorphism $f: M \to M$ and $h(v)$ is the (metric) entropy of the quotient endomorphism $f_\varepsilon$ induced by $f$ on the quotient space $M/\xi$; these two entropies are equal (see refs. 22 and 17).

**Corollary 6.4.**[8,12,17,29]

$$HD(\mu) = 1 - \frac{h(\mu)}{\lambda_2} \tag{6.3}$$

## ACKNOWLEDGMENTS

I wish to thank P. Collet for suggesting the existence of the above results and for his helpful advice.

## REFERENCES

1. R. Adler and B. Weiss, Similarity of automorphisms of the torus, *Mem. Am. Math. Soc.* **98**:1 (1969).
2. D. V. Anosov and Ya. G. Sinai, Some smooth ergodic systems, *Russ. Math. Surv.* **22**:103 (1967).
3. V. I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics* (Benjamin, 1968).
4. R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms* (Springer, 1975).
5. P. Collet, J. Lebowitz, and A. Porzio, The dimension spectrum of some dynamical systems, *J. Stat. Phys.* **47**:609 (1987).
6. J. P. Eckmann and I. Procaccia, Fluctuations of dynamical scaling indices in nonlinear systems, *Phys. Rev. A* **34**:659 (1986).
7. J. P. Eckmann and D. Ruelle, *Rev. Mod. Phys.* **57**:617 (1985).
8. P. Frederickson, J. L. Kaplan, E. D. Yorke, and J. A. Yorke, The Liapunov dimension of strange attractors, *J. Diff. Equations* **49**:105 (1983).
9. P. Grassberger, R. Badii, and A. Politi, Scaling laws for invariant measures on hyperbolic and nonhyperbolic attractors, *J. Stat. Phys.* **51**:135 (1988).
10. M. Gundlach, M. Sc. Dissertation, Warwick (1986).
11. M. Jensen, L. Kadanoff, J. Halsey, I. Procaccia, and B. Shraiman, Fractal measures and their singularities, the characterization of strange sets, *Phys. Rev. A* **33**:1141 (1986).
12. J. L. Kaplan and J. A. Yorke, Chaotic behavior of multidimensional difference equations, In *Lecture Notes in Mathematics*, Vol. 730 (1978), p. 228.
13. O. Lanford, *Qualitative and Statistical Theory of Dissipative Systems* (CIME Lectures, 1976).
14. O. Lanford, In *Lecture Notes in Physics*, Vol. 20 (1973), p. 1.
15. F. Ledrappier, Propriétés ergodiques des mesures de Sinai, *Pub. IHES* (1983), p. 163.
16. F. Ledrappier, Some relations between dimension and Liapunov exponents, *Commun. Math. Phys.* **81**:229 (1981).
17. F. Ledrappier and L. S. Young, The metric entropy of diffeomorphisms, *Ann. Math.* **122**:509 (1985).

18. Ya. B. Pesin, Characteristic Liapunov exponents and smooth ergodic theory, *Russ. Math. Surv.* **32**:55 (1977).
19. Ya. B. Pesin and Ya. G. Sinai, Gibbs measures for partially hyperbolic attractors, *Ergodic Theory Dyn. Syst.* **2**:417 (1982).
20. D. Rand, The singularities spectrum $f(\alpha)$ for cookie-cutters, Warwick preprint (1986).
21. V. A. Rholin, On the fundamental ideas of measure theory, *Am. Math. Soc. Trans.* **10**:1 (1962).
22. V. A. Rholin, Exact endomorphisms of a Lebesgue space, *Am. Math. Soc. Trans.* **39**:1 (1964).
23. D. Ruelle, A measure associated to Axiom A attractors, *Am. J. Math.* **98**:619 (1976).
24. D. Ruelle, Thermodynamic formalism, in *Encyclopedia of Mathematics*, Vol. 5 (Addison-Wesley, 1978).
25. D. Ruelle, *N. Y. Acad. Sci.* **357**:1 (1980).
26. S. Smale, Differentiable dynamical systems, *Bull. Am. Math. Soc.* **73**:747 (1967).
27. Ya. G. Sinai, Gibbs measures in ergodic theory, *Russ. Math. Surv.* **188**:21 (1972).
28. E. Vul, K. Khanin, and Ya. G. Sinai, Feigenbaum universality and thermodynamic formalism, *Russ. Math. Surv.* **39**:3 (1984).
29. L. S. Young, Dimension, entropy and Liapunov exponents, *Ergodic Theory Dyn. Syst.* **2**:109 (1982).

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